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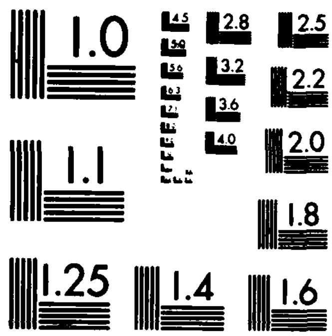
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Confidence Bands Under Proportional Hazards

by

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Asymptotic simultaneous confidence bands are derived for the survival function under the proportional hazards model of random right-censorship. These bands are based on the maximum likelihood estimator (MLE) of the survival function, rather than the well-known product limit estimator (PLE). In the case where the censoring parameter, denoted by β, is known the bands are asymptotically exact, while when β is unknown the bands are asymptotically conservative. For the case where β is unknown, the proposed bands are shown to be narrower than those proposed by Cheng and Chang (1985). Csörgő and Horváth's (1986) idea of mixing bands is then employed to obtain even narrower bands. As one would expect, under the more structured model, the PLE-based band of Gillespie and Fisher (1979) is shown to be inferior to the MLE-based bands, and this inferiority is more marked as the degree of censoring increases.

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Confidence Bands Under Proportional Hazards[1]

by

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Asymptotic simultaneous confidence bands are derived for the survival function under the proportional hazards model of random right-censorship. These bands are based on the maximum likelihood estimator ~~(MLE)~~ of the survival function, rather than the well-known product limit estimator (PLE). In the case where the censoring parameter, denoted by $\overset{\text{beta}}{\beta}$, is known the bands are asymptotically exact, while when $\overset{\text{beta}}{\beta}$ is unknown the bands are asymptotically conservative. For the case where $\overset{\text{beta}}{\beta}$ is unknown, the proposed bands are shown to be narrower than those proposed by Cheng and Chang (1985). Csörgő and Horváth's (1986) idea of mixing bands is then employed to obtain even narrower bands. As one would expect, under the more structured model, the PLE-based band of Gillespie and Fisher (1979) is shown to be inferior to the MLE-based bands, and this inferiority is more marked as the degree of censoring increases.

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Key Words and Phrases: Koziol-Green model, simultaneous confidence bands, weak convergence.

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1. Introduction and summary. Let (X_i, Y_i), i=1,...,n, be independent and identically distributed random vectors. In the proportional hazards model of random right-censorship, referred to in the sequel as the Koziol-Green (KG) model, X_i and Y_i are independent, the X_i's denote the survival times with continuous survival function $F(t)=P(X>t)$, and the Y_i's denote the censoring times with continuous censoring function $G(t)=P(Y>t)$. Furthermore, this model asserts that for some $\beta \geq 0$, possibly unknown, $G(t)=F(t)^\beta$ for all $t \geq 0$. The censored data consist of (Z_i, δ_i), i=1,...,n, where $Z_i = \text{minimum}(X_i, Y_i)$, $\delta_i = I(X_i < Y_i)$ and I(A) is the indicator of event A.

We develop asymptotic $(1-\alpha)100\%$ simultaneous confidence bands for F under the KG model. The bands are based on the maximum likelihood estimator (MLE) of F for this model. These bands, which are derived in Section 2, are presented in pairs with the bands in each pair being asymptotically equivalent. The bands in each pair are of the forms

(1.1) $\{[F_n(t)\{1 + r_n(t;\cdot)\}^{-1}, F_n(t)\{1 - r_n(t;\cdot)\}^{-1}], 0 \leq t \leq T\}$

and

(1.2) $\{[F_n(t)\{1 - r_n(t;\cdot)\}, F_n(t)\{1 + r_n(t;\cdot)\}], \ 0 \leq t \leq T\}$,

where $F_n(t)$ is the MLE of F given in (2.1), and $r_n(t;\cdot)$ depends on at least one tabular value, with the tabular values depending on β, α, and H(T), where H=FG. In the case where β is known the bands are asymptotically exact, while when β is unknown the bands are asymptotically conservative, though less conservative than bands proposed by Cheng and Chang (1985) under the same model. Csörgő and Horváth's (1986) idea of mixing bands is then employed to obtain narrower mixed bands. In Section 3 we assess the adequacy of the asymptotic results for finite sample sizes. For different values of n, β, and α, estimates of the achieved confidence levels for each of the bands are obtained through a computer simulation. In Section 4 the asymptotic widths of the bands are compared analytically. It is shown that when β is unknown, some of the bands presented here are narrower than those of Cheng and Chang (1985). Furthermore, when β is large and the KG model holds, the MLE-based mixed bands outperform the Gillespie and Fisher (1979) band based on the Kaplan and Meier (1958) product-limit estimator (PLE) of F.

The KG model has received considerable attention in the literature. This model was introduced by Koziol and Green (1976) in the context of developing a PLE-based goodness-of-fit test for censored data. Earlier, Efron (1967) used a special case of this model to compare the efficiencies of various two-sample tests for censored data. Under this model, Csörgő and Horváth (1981) presented a PLE-based confidence band and a goodness-of-fit test for F. Their goodness-of-fit test improved upon that of Koziol and Green (1976)in that their test does not require that β be known. Also, under the assumption that the KG model holds, Chen, Hollander and Langberg (1982) and Wellner (1985) obtained the exact moments of versions of the PLE, and studied the applicability of using the asymptotic variance of the PLE in place of its exact variance for finite samples. Emoto (1984) also utilized a special case of the KG model in comparing the mean-square errors of the PLE and the MLE of the survival function.

The importance of the KG model in reliability theory and survival analysis should not be underestimated. The model arises naturally in reliability studies, and in particular, in competing risks models (cf. Example 1 of Chen, Hollander and Langberg, 1982). More importantly, its great tractability enables the study of the performances and properties of other methods developed for more general models (such as the random right-censorship model where no structural relationship between F and G is assumed) in settings where these methods are not optimal. In this spirit, Chen, Hollander and Langberg (1982) and Wellner (1985) were able to study how well the asymptotic variance of the PLE approximates the exact variance when the KG model holds. In a similar vein, we are able to study the Gillespie and Fisher (1979) PLE-based band under the KG model where it has the correct asymptotic coverage probability but it is not the preferred band. In many ways, the KG model assumes a position in reliability models similar to the role that its precursor, the Lehmann alternatives, has in classical nonparametric settings such as the one- and two-sample problems. Lehmann's (1953) alternatives also arise naturally in certain situations, and therefore the tests derived for these alternatives are of importance in their own right. However such tests serve also as standards for assessing how much power other competing tests (such as the Wilcoxon or normal scores tests) sacrifice in models for which they are not optimal. The KG model and procedures derived under it fulfill a similar role in the domain of reliability models.

2. Confidence bands for the survival function. The maximum likelihood estimator of F under the KG model is given by

$$(2.1)\quad F_n(t) = \begin{cases} H_n(t)^{\gamma} & \text{if } \beta \text{ is known} \\ H_n(t)^{\gamma(n)} & \text{if } \beta \text{ is unknown,} \end{cases}$$

where $\gamma=(1+\beta)^{-1}$, $\gamma_n=\gamma(n)=n^{-1}\sum_{i=1}^{n}\delta_i$ and $H_n(t)=n^{-1}\sum_{i=1}^{n} I(Z_i>t)$. The parameter γ equals the probability of an uncensored observation, so β is referred to in the sequel as the censoring parameter. For ease of reference, the different pairs of bands presented below are labelled by A_n, B_n, ..., H_n, with the bands in each pair differentiated by a superscript of 1 or 2, e.g., A_n^1, A_n^2, etc., according to whether it is of the form (1.1) or (1.2), respectively. In (1.1) and (1.2) it will be implicit that $F_n(t)=H_n(t)^{\gamma}$ or $H_n(t)^{\gamma(n)}$ according to whether β is known or unknown, respectively. Furthermore, to facilitate the proofs of some results, we assume the existence of a basic probability space $(\Omega,\mathcal{F},P)$ such that the random functions discussed below are measurable maps from $(\Omega,\mathcal{F})$ to $(D,\mathcal{D})$, where the latter is Skorohod's measurable space on $[0,T]$ (cf. Billingsley, 1968, Chapter 3).

We first present the bands for the case where β is known. With minor modifications of Theorem 3 of Cheng and Lin (1984), the following weak convergence result for $F_n(t)=H_n(t)^\gamma$ is obtained.

Theorem 2.1: If $0<T<\infty$ with $F(T)>0$, then $W_n(t)=n^{1/2}[F_n(t)-F(t)] \Longrightarrow W(t)$ on $D[0,T]$, where $W(t)$ is a Gaussian process having zero mean and covariance function $Cov(W(s),W(t))=u(s)v(t)$ for $0\le s\le t\le T$, $u(s)=\gamma G(s)^{-1}[1-H(s)]$, $v(t)=\gamma G(t)^{-1}H(t)$, and "$\Longrightarrow$" denotes "converges weakly".

Let $d^{-1}(s)=\inf\{t: d(t)\ge s\}$ be the inverse of $d(t)=u(t)/v(t)=[1-H(t)]/H(t)$. Since F and G are continuous then $d^{-1}(s)$ is strictly increasing. By Theorem 2.1 and the transformation of Doob (1949) the following corollary is immediate.

Corollary 2.1: If $0<T<\infty$ with $F(T)>0$, then $W_n(t)/[\gamma F(t)] \Longrightarrow B(d(t))$ on $D[0,T]$, where $B(t)$ is the standard Brownian motion process.

Corollary 2.2: If $0<T<\infty$ with $F(T)>0$, then $W_n(t)/[\gamma F_n(t)] \Longrightarrow B(d(t))$ on $D[0,T]$.

Notation: In the remainder of the paper, $\sup_t$ will mean $\sup_{0\le t\le T}$ and $\lim_n$ will mean $\lim_{n\to\infty}$, except when otherwise stated.

Proof of Corollary 2.2: By Theorem 4.1 of Billingsley (1968 , p. 25) it suffices to show that $\sup_t|\{W_n(t)/F_n(t)\} - \{W_n(t)/F(t)\}| \to 0$ almost surely (a.s.). By the continuous mapping theorem (Billingsley, 1968, Theorem 5.1), $\sup_t|W_n(t)| \to \sup_t|W(t)|$ in distribution. Since $P\{\sup_t|W(t)|<\infty\}=1$, it therefore remains to show that $\sup_t|\{1/F_n(t)\} - \{1/F(t)\}| \to 0$ a.s.

Let $\varepsilon=F(T)>0$ and take $\omega_0\in\Omega_0=\{\omega\in\Omega: \sup_t|H_n(t) - H(t)| \to 0\}$. Then there exists an integer $N=N(\omega_0,\varepsilon)$ such that $H_n(t)\ge(\varepsilon/2)^{1/\gamma}$ for all $n\ge N$. Thus for this ω_0 and $n\ge N$, $\sup_t|\{1/F_n(t)\} - \{1/F(t)\}|\le 2\varepsilon^{-2}\sup_t|F_n(t) - F(t)|=2\varepsilon^{-2}\sup_t|H_n(t)^\gamma - H(t)^\gamma| \to 0$. Since ω_0 is arbitrary and $P(\Omega_0)=1$ by the uniform strong consistency of $H_n(t)$, the proof is complete. ||

Let $Q_T(\lambda_1,\lambda_2)$ denote the probability that the process $B(t)$ lies between the lines $-(\lambda_1 + \lambda_2 t)$ and $(\lambda_1 + \lambda_2 t)$ for all $0 \le t \le T$. From Anderson (1960), we find that for $\lambda_1 > 0$,

$$Q_T(\lambda_1,\lambda_2)=P\{|B(t)|\le\lambda_1 + \lambda_2 t,\ 0\le t\le T\}$$

$$=1-2\Phi[(-\lambda_2 T-\lambda_1)/T^{1/2}]-2\sum_{k=1}^{\infty}\exp(-2\lambda_1\lambda_2 k^2)\cdot \tag{2.2}$$

$$\{\Phi[(\lambda_2 T+2\lambda_1 k+\lambda_1)/T^{1/2}] - \Phi[(-\lambda_2 T+2\lambda_1 k-\lambda_1)/T^{1/2}]\},$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. Also, denote the empirical version of $d(t)$ by $d_n(t)=[1-H_n(t)]/H_n(t)$. Then the following theorem provides the $A_n(t;\lambda_1^a,\lambda_2^a)$ bands.

<u>Theorem 2.2:</u> Let $0<\alpha<1$ and $0<T<\infty$ with $F(T)>0$. Then the bands in (1.1) and (1.2) with $r_n(t;\lambda_1^a,\lambda_2^a)=n^{-1/2}\gamma[\lambda_1^a + \lambda_2^a d_n(t)]$ are asymptotic $(1-\alpha)100\%$ simultaneous confidence bands for F on $[0,T]$ whenever $Q_{d(T)}(\lambda_1^a,\lambda_2^a)=1-\alpha$.

<u>Proof:</u> By the choice of λ_1^a and λ_2^a, and Corollary 2.1,

$$1-\alpha=P\{|B(t)|\le\lambda_1^a + \lambda_2^a t,\ 0\le t\le d(T)\}$$

$$=P\{|B(d(t))|\le\lambda_1^a + \lambda_2^a d(t),\ 0\le t\le T\}$$

$$=\lim_n P\{|W_n(t)/[\gamma F(t)]|\le\lambda_1^a + \lambda_2^a d(t),\ 0\le t\le T\}. \tag{2.3}$$

A similar argument as in Corollary 2.2 shows that $\sup_t|d_n(t) - d(t)| \to 0$ a.s., so for arbitrary $\varepsilon>0$, (2.3) can be bounded from below and above by $\lim_n P\{|W_n(t)/[\gamma F(t)]|\le\lambda_1^a + \lambda_2^a[d_n(t) \pm \varepsilon],\ 0\le t\le T\}$. Since ε is arbitrary, it follows that $1-\alpha=\lim_n P\{|W_n(t)/[\gamma F(t)]|\le\lambda_1^a + \lambda_2^a d_n(t),\ 0\le t\le T\}$. Inverting the set inside the braces for $F(t)$ yields the band in (1.1) with $r_n(t;\lambda_1^a,\lambda_2^a)$ given in the theorem. A similar argument with Corollary 2.2 instead of Corollary 2.1 yields the band in (1.2). ||

Confidence bands for F are also derivable from transformations to the Brownian bridge process $B^0(t)$. In contrast to the PLE (cf. Hall and Wellner, 1980), the limiting process $W(t)$ in Theorem 2.1 can be transformed in two ways to $B^0(t)$.

Theorem 2.3: If $0<T<\infty$ with $F(T)>0$, then

(i) $\{B^0(L(t))\}_{0\le t\le T} \overset{d}{=} \{W(t)[1-L(t)]/F(t)\}_{0\le t\le T}$

(ii) $\{B^0(1-H(t))\}_{0\le t\le T} \overset{d}{=} \{W(t)H(t)/[\gamma F(t)]\}_{0\le t\le T}$,

where "$\overset{d}{=}$" means equal in distribution, and $L(t)=\gamma^2[1-H(t)]/\{H(t) + \gamma^2[1-H(t)]\}= \gamma^2 d(t)/[1 + \gamma^2 d(t)]$.

Cheng and Chang (1985) proved (i), and since the proof of (ii) is straightforward, it is omitted here.

Corollary 2.3: If $0<T<\infty$ with $F(T)>0$, then

(i) $W_n(t)[1-L(t)]/F(t) \Longrightarrow B^0(L(t))$ on $D[0,T]$,

(ii) $W_n(t)H(t)/[\gamma F(t)] \Longrightarrow B^0(1-H(t))$ on $D[0,T]$.

Proof: The results follow from Theorem 2.1 and Theorem 2.3. ||

Corollary 2.4: If $0<T<\infty$ with $F(T)>0$, then

(i) $W_n(t)[1-L(t)]/F_n(t) \Longrightarrow B^0(L(t))$ on $D[0,T]$,

(ii) $W_n(t)H(t)/[\gamma F_n(t)] \Longrightarrow B^0(1-H(t))$ on $D[0,T]$.

The proof of this corollary is analogous to that of Corollary 2.2 except that we use Corollary 2.3 instead of Corollary 2.1.

Crossing probabilities of the Brownian bridge process $B^0(t)$ have been well-studied in the literature (cf. Doob(1949), Anderson (1960), and Hall and Wellner (1980)). Here we only need to know that for $0<T<1$,

(2.4) $P\{\sup_t |B^0(t)|\le\lambda\}=Q_{T/(1-T)}(\lambda,\lambda)$.

We now present the $B_n(t;\lambda^b)$ and $C_n(t;\lambda^c)$ pairs of bands, respectively.

Theorem 2.4: Let $0<\alpha<1$ and $0<T<\infty$ with $F(T)>0$. Then the bands in (1.1) and (1.2) with

(i) $r_n(t;\lambda^b)=n^{-1/2}\lambda^b[1 + \gamma^2 d_n(t)]$,

(ii) $r_n(t;\lambda^c)=n^{-1/2}\lambda^c\gamma/H_n(t)$

are pairs of asymptotic $(1-\alpha)100\%$ simultaneous confidence bands for F on $[0,T]$ whenever $1-\alpha=Q_{\gamma^2 d(T)}(\lambda^b,\lambda^b)=Q_{d(T)}(\lambda^c,\lambda^c)$.

Proof. Cheng and Chang (1985) proved the result for band (1.2) with $r_n(t;\cdot)$ given by (i). The proof for band (1.1) with $r_n(t;\cdot)$ given by (i) is similar. Below, we just present the proof for the bands with $r_n(t;\cdot)$ given by (ii). By the choice of λ^c, we have

$$1-\alpha=Q_{d(T)}(\lambda^c,\lambda^c)=Q_{[1-H(T)]/H(T)}(\lambda^c,\lambda^c)$$

$$=P\{\sup_{0\le t\le 1-H(T)}|B^0(t)|\le\lambda^c\} \text{ (by 2.4)}$$

$$=P\{\sup_t|B^0(1-H(t))|\le\lambda^c\}$$

$$=\lim_n P\{\sup_t|W_n(t)H(t)/[\gamma F(t)]|\le\lambda^c\}$$

by Corollary 2.3(ii). Since γ_n and $H_n(t)$ are uniformly strongly consistent estimators of γ and $H(t)$, the above expression is

$$=\lim_n P\{\sup_t|W_n(t)H_n(t)/[\gamma_n F(t)]|\le\lambda^c\}.$$

Inverting the set inside the braces for F yields the band (1.1) with $r_n(t;\cdot)$ given by (ii). A similar argument using Corollary 2.4(ii) yields the other band. ||

In contrast with the case where β is known, it is difficult to construct asymptotically exact $(1-\alpha)100\%$ simultaneous confidence bands when β is unknown. The explanation for this is deferred until after the statement of Theorem 2.5. Cheng and Lin (1984) gave the following representation for $F_n(t)=H_n(t)^{\gamma(n)}$, the MLE of $F(t)$ when β is unknown:

$$(2.5)\quad Z_n(t)=n^{1/2}[F_n(t)-F(t)]=W_n(t)+V_n(t)+R_n(t)$$

where

$$W_n(t)=\gamma G(t)^{-1}n^{1/2}[H_n(t)-H(t)],\ V_n(t)=\gamma F(t)\{\ell n F(t)\}n^{1/2}(\gamma_n-\gamma),$$

with $\sup_t|R_n(t)|=0(\ell n\ell n(n)/n^{1/2})$ a.s. This representation, in conjunction with the functional and univariate central limit theorems, implies the following weak convergence result.

Theorem 2.5: If $0<T<\infty$ with $F(T)>0$, then $Z_n(t) \Longrightarrow Z(t)$, where $Z(t)$ is a Gaussian process having zero mean and covariance function $\mathrm{Cov}(Z(s),Z(t))=u(s)v(t) + w(s)w(t)$ for $0\le s\le t\le T$, where $u(s)$ and $v(t)$ are defined in Theorem 2.1, and $w(s)=\beta^{1/2}F(s)\ln F(s)$.

The form of the covariance function of the limiting process $Z(t)$ precludes the possibility of transforming Z into the Brownian motion process and/or the Brownian bridge process $B^0(t)$. This prevents us from obtaining exact asymptotic confidence bands for F through the use of the function $Q_T(\lambda_1,\lambda_2)$ in (2.2). An ambitious program of deriving crossing probabilities of $Z(t)$ might be possible, but the difficulty might far outweigh the benefits of such a program. We shall therefore content ourselves with asymptotically conservative bands.

From (2.5) it is clear that the limiting process can be represented by $Z(t)=W(t) + V(t)$ where $W(t)$ is defined in Theorem 2.1 and $V(t)$ is of the form

(2.6) $V(t)=\{\beta^{1/2}F(t)\ln F(t)\}N$,

where N is a standard normal random variable. Furthermore, since Z_i and δ_i are independent under the KG model (cf. Armitage (1959), Allen (1963), and Sethuraman (1965)), the limiting processes $W(t)$ and $V(t)$ are independent.

Theorem 2.6: Let $0<\alpha<1$ and $0<T<\infty$ with $F(T)>0$. Then the bands (1.1) and (1.2) with

(i) $r_n(t;\lambda^d,z^d)=n^{-1/2}\{\lambda^d[1 + \gamma_n^2 d_n(t)] + z^d\beta_n^{1/2}|\ln F_n(t)|\}$, and

(ii) $r_n(t;\lambda_1^e,\lambda_2^e,z^e)=n^{-1/2}\{\gamma_n[\lambda_1^e + \lambda_2^e d_n(t)] + z^e\beta_n^{1/2}|\ln F_n(t)|\}$

are pairs of conservative asymptotic $(1-\alpha)100\%$ confidence bands for F on $[0,T]$ whenever $1-\alpha=Q_{\gamma^2 d(T)}(\lambda^d,\lambda^d)[2\Phi(z^d)-1]=Q_{d(T)}(\lambda_1^e,\lambda_2^e)[2\Phi(z^e)-1]$, where $\beta_n=(1-\gamma_n)/\gamma_n$.

The resulting pairs of bands are respectively referred to as the $D_n(t;\lambda^d,z^d)$ and $E_n(t;\lambda_1^e,\lambda_2^e,z^e)$ bands. Band $D_n^2(t;\lambda^d,z^d)$ was introduced by Cheng and Chang (1985).

Below we present only the proof of band $E_n^1(t;\lambda_2^e,\lambda_2^e,z^e)$ since the proofs for the other bands are similar.

<u>Proof of Theorem 2.6:</u> By choice of λ_1^e, λ_2^e and z^e and by Corollary 2.1,

$$1-\alpha=P\{|W(t)/[\gamma F(t)]|\le\lambda_1^e + \lambda_2^e d(t),\ 0\le t\le T\}\cdot P\{|N|\le z^e\}$$

$$\le P\{|W(t)/[\gamma F(t)]| + |V(t)/[\gamma F(t)]|\le[\lambda_1^e + \lambda_2^e d(t)] + z^e|\beta^{1/2}(\beta+1)\ell nF(t)|,\ 0\le t\le T\}$$

$$\le P\{|Z(t)/[\gamma F(t)]|\le[\lambda_1^e + \lambda_2^e d(t)] + z^e|\beta^{1/2}(\beta+1)\ell nF(t)|,\ 0\le t\le T\}.$$

Using the strongly consistent estimators γ_n, $d_n(t)$, β_n and $F_n(t)$ for γ, $d(t)$, β and $F(t)$, respectively, and by Theorem 2.5, the preceding probability is

$$=\lim_n P\{|Z_n(t)/[\gamma_n F(t)]|\le[\lambda_1^e + \lambda_2^e d_n(t)] + z^e|\beta_n^{1/2}(\beta_n+1)\ell nF_n(t)|],\ 0\le t\le T\}.$$

Inverting the set inside the braces for $F(t)$ yields the band $E_n^1(t;\lambda_1^e,\lambda_2^e,z^e)$. ||

Notice that the first inequality in the proof is a very weak one. This makes the D_n-bands and E_n-bands extremely conservative. We could have obtained similar bands based on the Brownian bridge transformation in Corollary 2.3(ii) and Corollary 2.4(ii), but since these bands are extremely conservative we do not pursue them. Instead, the $F_n(t;\lambda^f)$ and $G_n(t;\lambda^g)$ pairs of bands will now be developed.

For $\lambda>0$ define the distribution functions $Q_T^{(1)}(\lambda)$ and $Q_T^{(2)}(\lambda)$ as

$$(2.7)\quad Q_T^{(1)}(\lambda)=[c_1(T)]^{-1}(2/\pi)^{1/2}\int_0^\lambda Q_{\gamma^2 d(T)}(\lambda-x,\lambda-x)\cdot \exp\{-(1/2)[x/c_1(T)]^2\}dx$$

and

$$(2.8)\quad Q_T^{(2)}(\lambda)=[c_2(T)]^{-1}(2/\pi)^{1/2}\int_0^\lambda Q_{d(T)}(\lambda-x,\lambda-x)\cdot \exp\{-(1/2)[x/c_2(T)]^2\}dx,$$

where

$$c_1(T)=\begin{cases}\beta^{1/2}\ \bar{L}(T)\,|\ln H(T)| & \text{if } \gamma^2+\gamma^2\ln H(T)+(1-\gamma^2)H(T)\geq 0\\ \beta^{1/2}\ \bar{L}(t_0)\,|\ln H(t_0)| & \text{if } \gamma^2+\gamma^2\ln H(T)+(1-\gamma^2)H(T)<0\end{cases}$$

with t_0 being the solution of the transcendental equation $\gamma^2+\gamma^2\ln H(t)+(1-\gamma^2)H(t)=0$, $\bar{L}(t)=1-L(t)$, and

$$c_2(T)=\begin{cases}\beta^{1/2}H(T)\,|\ln H(T)| & \text{if } H(T)\geq\exp(-1)\\ \beta^{1/2}\exp(-1) & \text{if } H(T)<\exp(-1).\end{cases}$$

<u>Theorem 2.7:</u> Let $0<\alpha<1$ and $0<T<\infty$ with $F(T)>0$. Then the bands (1.1) and (1.2) with

(i) $r_n(t;\lambda^f)=n^{-1/2}\lambda^f[1+\gamma_n^2 d_n(t)]$, and

(ii) $r_n(t;\lambda^g)=n^{-1/2}\lambda^g\gamma_n/H_n(t)$

are pairs of conservative asymptotic $(1-\alpha)100\%$ simultaneous confidence bands for F on $[0,T]$ whenever $1-\alpha=Q_T^{(1)}(\lambda^f)=Q_T^{(2)}(\lambda^g)$.

<u>Proof:</u> We present only the proof for band (1.1) with $r_n(t;\cdot)$ given in (i). The proofs of the others are analogous. By Theorem 2.5, we have

$$\begin{aligned}\lim_n P\{\sup_t|Z_n(t)\bar{L}(t)/F(t)|\leq\lambda^f\}&=P\{\sup_t|Z(t)\bar{L}(t)/F(t)|\leq\lambda^f\}\\ &=P\{\sup_t|W(t)\bar{L}(t)/F(t)\ +\ V(t)\bar{L}(t)/F(t)|\leq\lambda^f\}\\ &\geq P\{\sup_t|W(t)\bar{L}(t)/F(t)|\ +\ \sup_t|V(t)\bar{L}(t)/F(t)|\leq\lambda^f\}\\ &=\int_0^{\lambda^f}P\{\sup_t|B^0(L(t))|\leq\lambda^f-x\}dP\{\sup_t|V(t)\bar{L}(t)/F(t)|\leq x\}\end{aligned}$$

by Theorem 2.3(i) and the Convolution Theorem. But by (2.6), $\{\sup_t|V(t)\bar{L}(t)/F(t)|\leq x\}=\{|N|\leq x/[\sup_t|\beta^{1/2}\bar{L}(t)\ln H(t)|]\}$, and since $c_1(T)=\sup_t|\beta^{1/2}\bar{L}(t)\ln H(t)|$, then

$$\lim_n P\{\sup_t|Z_n(t)\bar{L}(t)/F(t)|\leq\lambda^f\}=\int_0^{\lambda^f}Q_{L(T)/[1-L(T)]}(\lambda^f-x,\lambda^f-x)\,dP\{|N|\leq x/c_1(T)\}$$

which by the choice of λ^f and the fact that $L(T)/[1-L(T)]=\gamma^2 d(T)$ equals $Q_T^{(1)}(\lambda^f)=1-\alpha$. Substituting the uniformly strongly consistent estimator $\bar{L}_n(t)=[1+\gamma_n^2 d_n(t)]^{-1}$ for $\bar{L}(t)$, and then inverting the set $\{\sup_t|Z_n(t)\bar{L}(t)/F(t)|\leq\lambda^f\}$ for $F(t)$ yields the band $F_n^1(t;\lambda^f)$. ||

The pairs of bands in Theorem 2.7 are respectively referred to as the $F_n(t;\lambda^f)$ and $G_n(t;\lambda^g)$ bands. To compute the constant $c_1(T)$ in the distribution $Q_T^{(1)}(\lambda)$ in (2.7) a transcendental equation has to be solved. However, since the bands are asymptotically conservative, we might lessen this conservatism by using $c_1^*(T)$ instead of $c_1(T)$, where the former is defined as

$$c_1^*(T)=\begin{cases}\beta^{1/2}\Gamma(T)\,|\ell nH(T)| & \text{if } \gamma^2+\gamma^2\ell nH(T)+(1-\gamma^2)H(T)\geq 0\\ [1+\gamma^2(e-1)]^{-1} & \text{if } \gamma^2+\gamma^2\ell nH(T)+(1-\gamma^2)H(T)<0.\end{cases}$$

Note here that since $c_1^*(T)\leq c_1(T)$ the tabular value λ^f obtained using $c_1^*(T)$ will be at most the tabular value obtained using $c_1(T)$.

To facilitate the use of the $F_n(t;\lambda^f)$ and $G_n(t;\lambda^g)$ bands, selected percentiles for the distributions $Q_T^{(1)}(\lambda)$ and $Q_T^{(2)}(\lambda)$ are given in Tables 1 and 2, respectively. In computing the percentiles of $Q_T^{(1)}(\lambda)$ in Table 1, $c_1^*(T)$ was used in place of $c_1(T)$. The IMSL routine DCADRE was employed to perform the numerical integration in (2.7) and (2.8). To use Tables 1 and 2, the user needs to know the values of β and $F(T)$, or at least the values of β_n and $F_n(T)$, the latter quantities being the strongly consistent estimators of β and $F(T)$, respectively. This technique of substituting empirical versions in place of unknown quantities is also required for determining the tabular values of the other bands. Clearly, the finite sample properties of the bands will be affected by substituting these empirical versions, but the asymptotic properties will remain unchanged.

By examining the upper and lower contours of the bands in each pair, one notes that the lower and upper contours of band (1.1) are always above the corresponding contours of band (1.2), with the relationship for the upper contours holding whenever $1-r_n(t;\cdot)>0$. We could therefore adapt the idea of Csörgő and Horváth (1986) of mixing the bands in each pair to obtain a narrower band. For example, we could define

$$A_n^*(t;\lambda_1^a,\lambda_2^a)=A_n^1(t;\lambda_1^a,\lambda_2^a)\cap A_n^2(t;\lambda_1^a,\lambda_2^a)$$

Table 1. Selected Percentile Points for the Distribution $Q_T^{(1)}(\lambda)$ for Some Values of β and F(T)

β	$1-\alpha$	F(T) 0.10	0.20	0.30	0.40	0.50	0.60
0.20	0.90	1.38	1.38	1.37	1.38	1.30	1.19
	0.95	1.52	1.52	1.51	1.53	1.45	1.33
	0.99	1.80	1.80	1.79	1.83	1.74	1.60
0.40	0.90	1.46	1.46	1.44	1.41	1.26	1.16
	0.95	1.61	1.61	1.59	1.56	1.41	1.30
	0.99	1.90	1.90	1.89	1.85	1.69	1.57
0.60	0.90	1.52	1.52	1.50	1.46	1.21	1.12
	0.95	1.67	1.67	1.66	1.62	1.35	1.25
	0.99	1.97	1.97	1.96	1.93	1.63	1.52
0.80	0.90	1.57	1.57	1.55	1.51	1.17	1.07
	0.95	1.72	1.72	1.71	1.67	1.31	1.20
	0.99	2.03	2.03	2.02	1.98	1.59	1.47
1.00	0.90	1.60	1.60	1.58	1.54	1.14	1.04
	0.95	1.76	1.76	1.75	1.70	1.28	1.17
	0.99	2.08	2.07	2.07	2.03	1.56	1.43
1.20	0.90	1.62	1.62	1.61	1.57	1.13	1.01
	0.95	1.78	1.78	1.77	1.73	1.26	1.14
	0.99	2.11	2.11	2.10	2.06	1.54	1.40
1.40	0.90	1.64	1.64	1.63	1.58	1.11	0.99
	0.95	1.80	1.80	1.79	1.75	1.25	1.12
	0.99	2.13	2.13	2.12	2.09	1.53	1.38
1.60	0.90	1.65	1.65	1.64	1.60	1.11	0.98
	0.95	1.82	1.82	1.81	1.77	1.26	1.11
	0.99	2.15	2.15	2.14	2.11	1.52	1.36
1.80	0.90	1.66	1.66	1.65	1.61	1.11	0.97
	0.95	1.82	1.82	1.82	1.78	1.24	1.10
	0.99	2.16	2.16	2.15	2.12	1.52	1.35
2.00	0.90	1.66	1.66	1.66	1.62	1.53	0.96
	0.95	1.83	1.83	1.82	1.79	1.70	1.09
	0.99	2.16	2.16	2.16	2.13	2.03	1.34

Table 2. Selected Percentile Points for the Distribution $Q_T^{(2)}(\lambda)$ for Some Values of β and F(T)

β	$1-\alpha$	F(T) 0.10	0.20	0.30	0.40	0.50	0.60
0.20	0.90	1.38	1.38	1.37	1.35	1.32	1.24
	0.95	1.52	1.51	1.51	1.50	1.46	1.38
	0.99	1.79	1.79	1.79	1.78	1.75	1.67
0.40	0.90	1.45	1.45	1.45	1.44	1.42	1.35
	0.95	1.60	1.60	1.60	1.59	1.56	1.50
	0.99	1.88	1.88	1.88	1.88	1.86	1.80
0.60	0.90	1.52	1.52	1.52	1.51	1.49	1.45
	0.95	1.67	1.67	1.66	1.66	1.65	1.60
	0.99	1.96	1.96	1.96	1.96	1.95	1.91
0.80	0.90	1.57	1.57	1.57	1.57	1.56	1.52
	0.95	1.73	1.73	1.73	1.73	1.72	1.68
	0.99	2.04	2.04	2.04	2.04	2.03	2.00
1.00	0.90	1.62	1.62	1.62	1.62	1.61	1.59
	0.95	1.79	1.79	1.79	1.78	1.78	1.76
	0.99	2.11	2.11	2.11	2.11	2.10	2.08
1.20	0.90	1.67	1.67	1.67	1.67	1.66	1.65
	0.95	1.84	1.84	1.84	1.84	1.83	1.82
	0.99	2.17	2.17	2.17	2.17	2.17	2.16
1.40	0.90	1.72	1.72	1.72	1.72	1.72	1.70
	0.95	1.89	1.89	1.89	1.89	1.89	1.87
	0.99	2.24	2.24	2.24	2.24	2.24	2.23
1.60	0.90	1.76	1.76	1.76	1.76	1.76	1.74
	0.95	1.94	1.94	1.94	1.94	1.94	1.93
	0.99	2.30	2.30	2.30	2.30	2.30	2.29
1.80	0.90	1.80	1.80	1.80	1.80	1.80	1.79
	0.95	1.99	1.99	1.99	1.99	1.98	1.98
	0.99	2.36	2.36	2.36	2.36	2.36	2.35
2.00	0.90	1.84	1.84	1.84	1.84	1.84	1.83
	0.95	2.03	2.03	2.03	2.03	2.03	2.02
	0.99	2.42	2.42	2.42	2.42	2.41	2.41

as the A_n-mixed band. One must however be cautious in using such mixed bands since as pointed out in Csörgő and Horváth (1986), the finite sample coverage probability might be nowhere from the nominal asymptotic coverage probability of $1-\alpha$ if the bands being mixed are appreciably different. This idea of mixing bands to obtain narrower ones is however very appealing for the case where β is unknown, since the bands being mixed are asymptotically conservative and the mixed bands might still turn out to have finite sample coverage probabilities of at least $1-\alpha$.

Our comparisons in Sections 3 and 4 include the PLE-based GF-type bands (cf. Gillespie and Fisher, 1979 and Csörgő and Horváth, 1986). This pair, labelled the $H_n(t;\lambda_1^h,\lambda_2^h)$ bands, are given by

(2.9) $$H_n^1(t;\lambda_1^h,\lambda_2^h)=\{[S_n(t)\{1 + r_n(t;\lambda_1^h,\lambda_2^h)\}^{-1}, S_n(t)\{1 - r_n(t;\lambda_1^h,\lambda_2^h)\}^{-1}],\ 0\le t\le T\}$$

and

(2.10) $$H_n^2(t;\lambda_1^h,\lambda_2^h)=\{[S_n(t)\{1 - r_n(t;\lambda_1^h,\lambda_2^h)\}, S_n(t)\{1 + r_n(t;\lambda_1^h,\lambda_2^h)\}],\ 0\le t\le T\},$$

where $r_n(t;\lambda_1^h,\lambda_2^h)=n^{-1/2}[\lambda_1^h + \lambda_2^h d_n^*(t)]$,

$$Q_{\gamma d(T)}(\lambda_1^h,\lambda_2^h) = 1-\alpha,$$

$$d_n^*(t)=-\int_0^t [S_n(u)H_n(u)]^{-1}dS_n(u),$$

and $S_n(t)$ is the PLE of F. The PLE is defined as

$$S_n(t)=\Pi_{\{i:Z_i\le t\}}[(n-R_i)/(n-R_i+1)]^{\delta_i} I(t<\max_{1\le i\le n}Z_i),$$

where R_i is the rank of Z_i in the lexicographic ordering of (Z_i,δ_i), $i=1,\dots,n$, and with the convention that the product over an empty set equals 1.

3. Adequacy of the asymptotic results for finite samples. To assess the adequacy of the asymptotic results for finite sample sizes, a computer simulation was performed to obtain estimates of the error probabilities of the bands A_n^2, ..., H_n^2.

For each combination of n, β and α, 500 randomly censored samples were generated via the KG model with F exponential with scale parameter 1 and G exponential with scale parameter β. For each of these 500 samples, the asymptotic $(1-\alpha)100\%$ simultaneous confidence bands $A_n^2, \ldots, H_n^2$ for F on the interval [0,1] were constructed. The proportion among the 500 bands that did not contain F(t) on the interval [0,1] was then determined for each of the bands $A_n^2, \ldots, H_n^2$. The values of n were set to 50, 100 and 200, while the asymptotic error levels α were set to 0.01, 0.05 and 0.10. The censoring parameter β took the values 0.5, 1.0 and 2.0, which amounted to 33%, 50% and 67% censoring, respectively. This simulation was performed on a Cyber 730 computer at the Florida State University Computer Center. The uniform random number generator used was the intrinsic routine RANF, and a result by Lurie and Hartley (1972) concerning the sequential generation of ordered samples without recourse to sorting was employed.

Table 3 is a summary of the asymptotic tabular values [that is, in computing the tabular values we used the known values of β, F(t) and G(t)] utilized in constructing the bands for the different values of β and α. We imposed the restriction $\lambda_1^e=\lambda_2^e$ and $\lambda_1^h=\lambda_2^h$, while the tabular values λ_1^a and λ_2^a were chosen so the asymptotic widths of $A_n^2(t;\lambda_1^a,\lambda_2^a)$ at t=0 and t=1/2 were approximately equal.

Tables 4a-c are summaries of the observed error probabilities. Based on these results, the following rough conclusions can be made:

(i) When the censoring proportion is small (β=0.50), the asymptotic error level serves as an acceptable approximation to the true error level of the bands for moderate sample sizes. This conclusion is supported by the fact that the observed error levels of the asymptotically exact MLE-based bands A_n^2, B_n^2 and C_n^2, and the PLE-based band H_n^2 are close to α. However, as β increases from 0.5 to 1.0 and 2.0, the observed error levels for n=50 are far from α although they do get closer to α when n is increased. Thus, generally, the more censoring

Table 3. Tabular Values Used in Computer Simulation for Different Values of β and α with $T = 1.0$

β	α	Bands										
		A_n^2		B_n^2	C_n^2	D_n^2		E_n^2		F_n^2	G_n^2	H_n^2
		λ_1^a	λ_2^a	λ^b	λ^c	λ^d	z^d	$\lambda_1^e=\lambda_2^e$	z^e	λ^f	λ^g	$\lambda_1^h=\lambda_2^h$
0.5	0.01	2.1	1.25	1.60	1.63	1.71	2.81	1.73	2.81	1.91	1.92	1.62
	0.05	1.8	1.00	1.32	1.36	1.45	2.24	1.48	2.24	1.61	1.63	1.35
	0.10	1.6	0.90	1.18	1.22	1.32	1.95	1.35	1.95	1.45	1.48	1.21
1.0	0.01	2.6	1.01	1.60	1.63	1.71	2.81	1.73	2.81	2.05	2.11	1.63
	0.05	2.2	0.83	1.33	1.36	1.45	2.24	1.48	2.24	1.72	1.78	1.35
	0.10	2.0	0.73	1.19	1.22	1.32	1.95	1.35	1.95	1.56	1.62	1.22
2.0	0.01	3.8	0.70	1.62	1.63	1.72	2.81	1.73	2.81	2.15	2.42	1.63
	0.05	3.1	0.59	1.34	1.36	1.47	2.24	1.48	2.24	1.81	2.03	1.36
	0.10	2.8	0.53	1.21	1.22	1.34	1.95	1.35	1.95	1.64	1.84	1.22

Table 4a. Observed Error Probabilities in Simulation for the Case $\beta = 0.5$ -- 33% Censoring

α	n	Bands							
		A_n^2	B_n^2	C_n^2	D_n^2	E_n^2	F_n^2	G_n^2	H_n^2
0.01	50	.008	.010	.006	0	0	.002	.002	.008
	100	.004	.008	.004	0	0	.004	0	.006
	200	.006	.012	.006	0	0	.002	.002	.004
0.05	50	.036	.054	.030	.004	0	.018	.018	.046
	100	.058	.070	.044	.002	.002	.024	.014	.050
	200	.046	.050	.042	.008	.004	.020	.024	.040
0.10	50	.100	.110	.078	.008	.006	.046	.026	.088
	100	.116	.116	.086	.018	.012	.064	.036	.098
	200	.108	.110	.106	.008	.012	.052	.052	.098

Table 4b. Observed Error Probabilities in Simulation for the Case $\beta = 1.0$ -- 50% Censoring

α	n	Bands							
		A_n^2	B_n^2	C_n^2	D_n^2	E_n^2	F_n^2	G_n^2	H_n^2
0.01	50	.004	.008	.004	0	.002	.002	.006	.008
	100	.004	.012	.006	0	.002	.002	.010	.010
	200	.012	.014	.006	0	0	.006	.006	.004
0.05	50	.022	.026	.026	.002	.004	.018	.028	.034
	100	.030	.034	.032	0	0	.012	.014	.016
	200	.046	.052	.026	.004	.002	.036	.024	.052
0.10	50	.056	.078	.044	0	.006	.024	.022	.058
	100	.090	.098	.072	.010	.008	.042	.026	.060
	200	.094	.112	.068	.018	.002	.060	.034	.100

Table 4c. Observed Error Probabilities in Simulation for the Case $\beta = 2.0$ -- 67% Censoring

α	n	Bands							
		A_n^2	B_n^2	C_n^2	D_n^2	E_n^2	F_n^2	G_n^2	H_n^2
0.01	50	.002	0	.002	0	0	.004	.014	.024
	100	.002	.004	.002	0	0	.002	.006	.014
	200	0	.002	.002	0	0	.006	.002	.014
0.05	50	.016	.014	.016	0	.008	.016	.028	.062
	100	.016	.024	.020	0	.006	.026	.024	.046
	200	.036	.032	.036	.002	.002	.034	.014	.076
0.10	50	.038	.034	.044	0	.010	.052	.044	.080
	100	.076	.080	.062	.006	.010	.070	.024	.068
	200	.084	.094	.106	0	.016	.070	.040	.102

there is, the larger the sample needed for the asymptotic levels to serve as good approximations to the finite sample error levels of the bands.

(ii) The bands $D_n^2(t;\lambda^d,z^d)$ and $D_n^2(t;\lambda^e,\lambda^e,z^e)$ are extremely conservative, while $F_n^2(t;\lambda^f)$ and $G_n^2(t;\lambda^g)$ are less conservative than D_n^2 and E_n^2, with F_n^2 tending to be less conservative than G_n^2.

4. Comparison of widths of the bands. In this section we present analytic comparisons of the asymptotic widths of the bands, and determine regions where each tends to perform well. Let $\omega_n(t;\cdot)$ denote the width of the bands (1.2) and (2.10), and define the asymptotic width to be $\lim_{n\to\infty} n^{1/2}\omega_n(t;\cdot)$. To distinguish which pair of bands is referred to, let

(4.1) $\chi_\theta(t;\cdot)=\lim_n n^{1/2}\omega_n(t;\cdot)=2F(t)\lim_n n^{1/2} r_n(t;\cdot)$ a.s.,

where θ is the letter label of the pair of band, e.g., $\chi_A(t;\lambda_1^a,\lambda_2^a)$ is the asymptotic width of pair A_n. In (4.1) we used the strong consistency of $F_n(t)$ and $S_n(t)$ for $F(t)$, and the fact that the asymptotic widths of the bands in (1.1) and (1.2), and in (2.9) and (2.10) are equal. Table 5 is a summary of the asymptotic widths of the different pairs of bands, and the conditions that their corresponding tabular values must satisfy. The asymptotic width of the PLE-based H_n-band is obtained by noting that

$$d_n^*(t)=-\int_0^t [S_n(u)H_n(u)]^{-1}dS_n(u)\longrightarrow \gamma d(t) \text{ a.s.}$$

when the KG model obtains.

To simplify our comparison below we assume that $\lambda_1^a=\lambda_2^a=\lambda^a$, $\lambda_1^e=\lambda_2^e=\lambda^e$ and $\lambda_1^h=\lambda_2^h=\lambda^h$. Under the first assumption, band C_n is identical to band A_n. We now compare the asymptotic widths of the bands for the case where β is known.

Since $d(T)\geq\gamma d(T)\geq\gamma^2 d(T)$ it follows from (2.2) that $\lambda^a\geq\lambda^h\geq\lambda^b$. This immediately implies that $\chi_B(t;\lambda^b)\leq\chi_H(t;\lambda^h,\lambda^h)$ for $0\leq t\leq T$, and the difference in asymptotic width is $2F(t)\{(\lambda^h-\lambda^b) + \gamma d(t)(\lambda^h-\gamma\lambda^b)\}$ which gets large for large t.

The effect on this difference due to a change in β is not very clear since λ^b and λ^h both depend on β. Nevertheless, by acting as if $\lambda^b \cong \lambda^h$, which is indicated by Table 3, the maximum difference occurs at approximately $\beta=1$ and decreases as β becomes different from 1 in either direction. On the other hand, the difference in asymptotic widths of bands H_n and A_n is $2F(t)\{(\lambda^h-\lambda^a\gamma) + \gamma d(t)(\lambda^h-\lambda^a)\}$, and since Table 3 also indicates that $\lambda^h \cong \lambda^a = \lambda^c$, this difference is positive and increases with either an increase in β or a decrease in t. Thus, both MLE-based bands outperform the Gillespie-Fisher PLE-based band in terms of their asymptotic widths.

It is a simple algebraic exercise to show that

$$\chi_A(t;\lambda^a,\lambda^a) \le \chi_B(t;\lambda^b) \iff t \le H^{-1}\{\gamma(\lambda^a-\gamma\lambda^b)/[\lambda^b(1-\gamma^2)]\}.$$

Since $\lambda^a \ge \lambda^b$ it follows that $H^{-1}\{\gamma(\lambda^a-\gamma\lambda^b)/[\lambda^b(1-\gamma^2)]\} \le H^{-1}\{\gamma/(1+\gamma)\} = F^{-1}\{[\gamma/(1+\gamma)]^\gamma\}$. Furthermore, since $\{\gamma/(1+\gamma)\}^\gamma$ decreases rapidly from 1.0 to 0.5 as β increases from 0 to 1, we deduce that the interval $\{t: \chi_A(t;\lambda^a,\lambda^a) \le \chi_B(t;\lambda^b)$ gets shorter as β increases, but does so at a slow rate. Adapting again the idea of a mixed band, we can define the band

$$(AB)_n(t;\lambda^a,\lambda^a,\lambda^b) = A_n(t;\lambda^a,\lambda^a) \cap B_n(t;\lambda^b),$$

which would still be an asymptotically $(1-\alpha)100\%$ simultaneous confidence band for F, but whose finite sample coverage probability might be much less than $1-\alpha$.

Similar results carry over to the case where β is unknown. Setting $z^d=z^e$, we have

$$\chi_D(t;\lambda^d,z^d) \ge \chi_E(t;\lambda^e,\lambda^e,z^e) \iff t \le H^{-1}\{\gamma(\lambda^e-\lambda^d\gamma)/[\lambda^d(1-\gamma^2)]\},$$

and

$$\chi_F(t;\lambda^f) \ge \chi_G(t;\lambda^g) \iff t \le H^{-1}\{\gamma(\lambda^g-\lambda^f\gamma)/[\lambda^f(1-\gamma^2)]\}.$$

Thus the E_n– and G_n–bands are narrower for small t, while the D_n– and F_n–bands are narrower for large t. From the derivation of the bands in Section 2 and the simulation results of Section 3 we have seen that the D_n– and E_n–bands are extremely conservative. Intuitively, we therefore expect them to be wider than

Table 5. Asymptotic Widths of the Pairs of Bands A_n to H_n with Corresponding Requirements for Their Tabular Values

Band, θ	$[2F(t)]^{-1}X_\theta(t;\cdot)$	Requirements for Tabular Values
A	$\gamma[\lambda_1^a+\lambda_2^a d(t)]$	$Q_{d(T)}(\lambda_1^a,\lambda_2^a)=1-\alpha$
B	$\lambda^b[1+\gamma^2 d(t)]$	$Q_{\gamma^2 d(T)}(\lambda^b,\lambda^b)=1-\alpha$
C	$\lambda^c\gamma/H(t)$	$Q_{d(T)}(\lambda^c,\lambda^c)=1-\alpha$
D	$\lambda^d[1+\gamma^2 d(t)]+z^d\beta^{1/2}\|\ln F(t)\|$	$Q_{\gamma^2 d(T)}(\lambda^d,\lambda^d)[2\Phi(z^d)-1]=1-\alpha$
E	$\gamma[\lambda_1^e+\lambda_2^e d(t)]+z^e\beta^{1/2}\|\ln F(t)\|$	$Q_{d(T)}(\lambda_1^e,\lambda_2^e)[2\Phi(z^e)-1]=1-\alpha$
F	$\lambda^f[1+\gamma^2 d(t)]$	$Q_T^{(1)}(\lambda^f)=1-\alpha$
G	$\lambda^g\gamma/H(t)$	$Q_T^{(2)}(\lambda^g)=1-\alpha$
H	$[\lambda_1^h+\lambda_2^h\gamma d(t)]$	$Q_{\gamma d(T)}(\lambda_1^h,\lambda_2^h)=1-\alpha$

the less conservative bands F_n and G_n. We show this below. From Table 5 it is easy to verify that $\chi_E(t;\lambda^e,\lambda^e,z^e)\geq\chi_G(t;\lambda^g)$ if and only if t satisfies the transcendental equation

(4.2) $\quad H(t)^{H(t)}\leq\exp\{-(\lambda^g-\lambda^e)/(z^e\beta^{1/2})\}.$

The function in the left side of (4.2), which is graphed in Figure 1, has a minimum value of 0.6922 at $H(t)=\exp(-1)$, and drops rapidly from 1.0 and slowly climbs back to 1.0 as $H(t)$ varies from 0 to 1. This is the reason why the interval $\{t:\chi_E(t;\lambda^e,\lambda^e,z^e)\geq\chi_G(t;\lambda^g)\}$ is wide. For example, using the tabular values in Table 3 and referring to Figure 1, the G_n- band is narrower than the E_n—band in the approximate intervals (0.07,2.46), (0.08,1.55) and (0.07,0.93) for β=0.5, 1.0 and 2.0, respectively. Similarly, $\chi_D(t;\lambda^d,z^d)\geq\chi_F(t;\lambda^f)$ whenever t satisfies

$$H(t)^{H(t)}\leq\exp\{-(\lambda^f-\lambda^d)/[z^d\beta^{1/2}(\beta+1)]\}.$$

Referring again to Figure 1 and using the tabular values in Table 3, we find (0.05,2.71), (0.03,2.13) and (0.02,1.64) to be the approximate intervals where band F_n is narrower than band D_n for β=0.5, 1.0 and 2.0, respectively. For all practical purposes, we can therefore consider the F_n- and G_n- bands to be narrower than the D_n— and E_n—bands in the regions where they perform well. As in the case where β is known, we could also define the mixed band

$$(FG)_n(t;\lambda^f,\lambda^g) = F_n(t;\lambda^f) \cap G_n(t;\lambda^g)$$

which will be narrower on almost the whole region of interest as compared to the other bands presented. The simulation results in Section 3 indicate that band $(FG)_n$ may still be conservative.

The PLE-based GF-type H_n-band is wider than the F_n- and G_n-bands whenever $t\geq H^{-1}(\eta)$ and $t<H^{-1}(\xi)$, respectively, where $\eta=\{\gamma(\lambda^f\gamma-\lambda^h)\}/\{(1-\gamma)[\lambda^h-\lambda^f(1+\gamma)]\}$ and $\xi=\{\gamma(\lambda^g-\lambda^h)\}/\{\lambda^h(1-\gamma)\}$. To get some idea on these changeover points, notice that except for the case β=0.5 and α=0.10 in Table 3, $H^{-1}(\eta)\leq H^{-1}(\xi)$. Thus, in

FIGURE 1. Graph of H(t) versus $H(t)^{H(t)}$

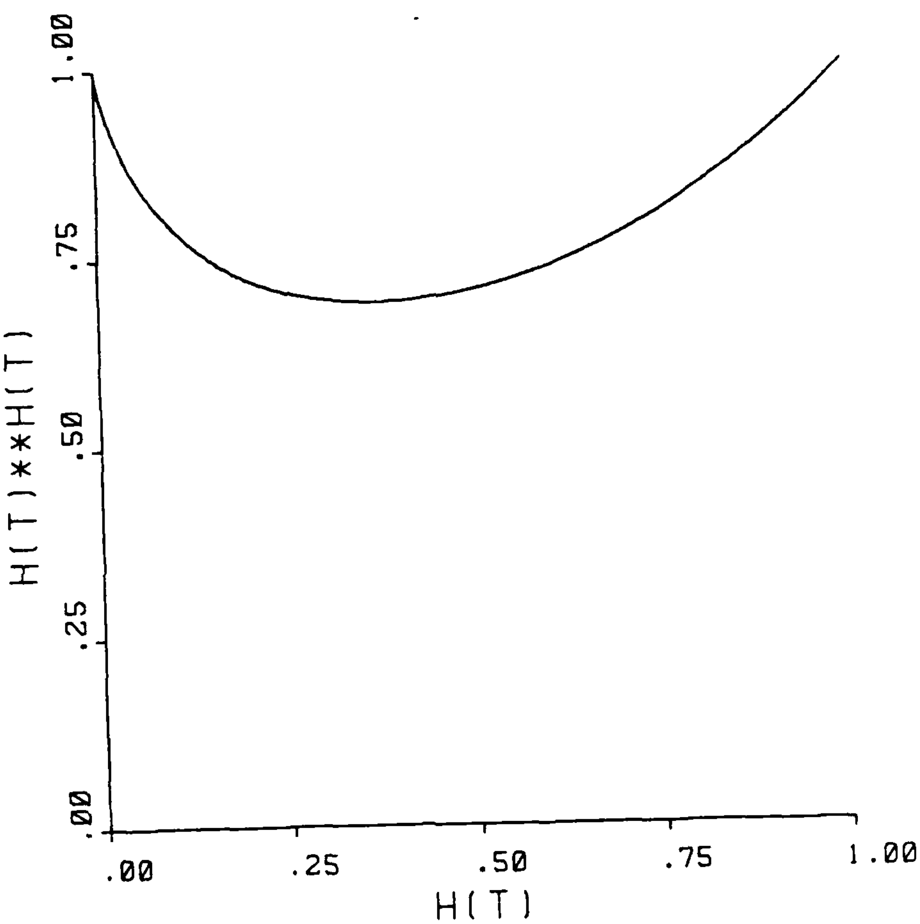


Figure 2a. Plots of 90% Confidence Bands for Randomly Generated Data Under the KG Model with $\beta = 0.5$

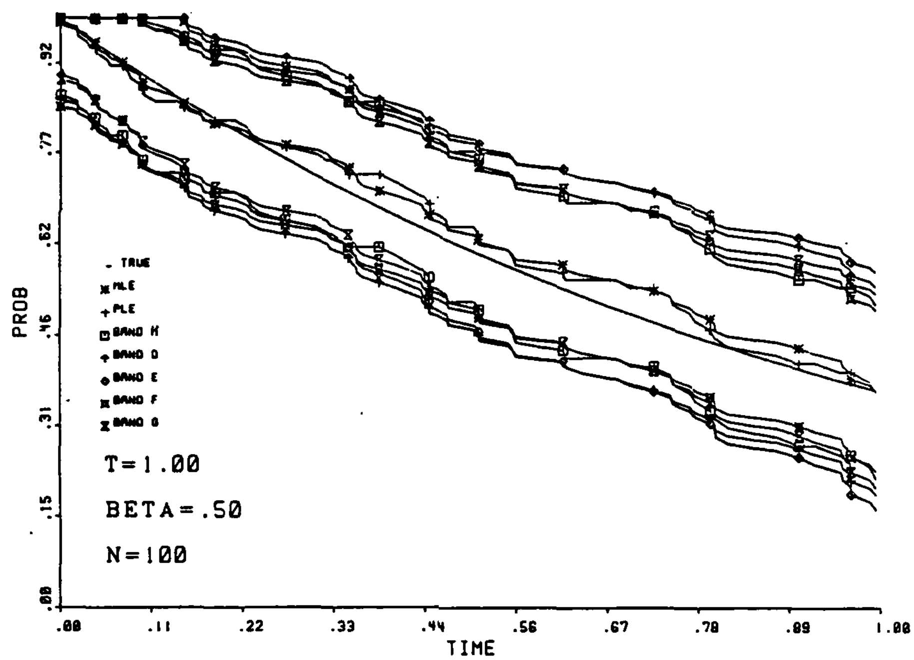


Figure 2b. Plots of 90% Confidence Bands for Randomly Generated Data Under the KG Model with $\beta = 1.0$

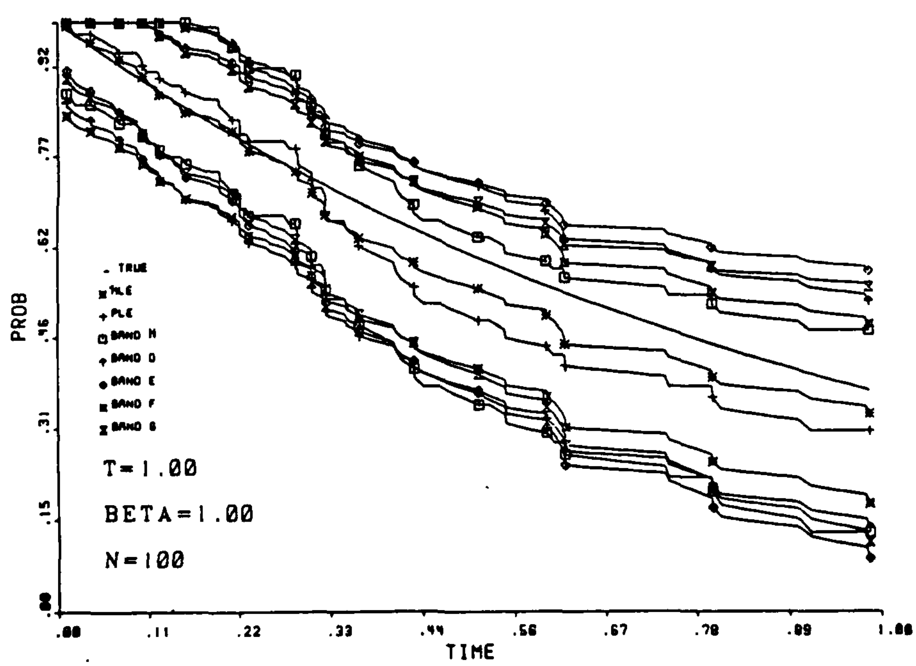


Figure 2c. Plots of 90% Confidence Bands for Randomly Generated Data Under the KG Model with $\beta = 1.5$

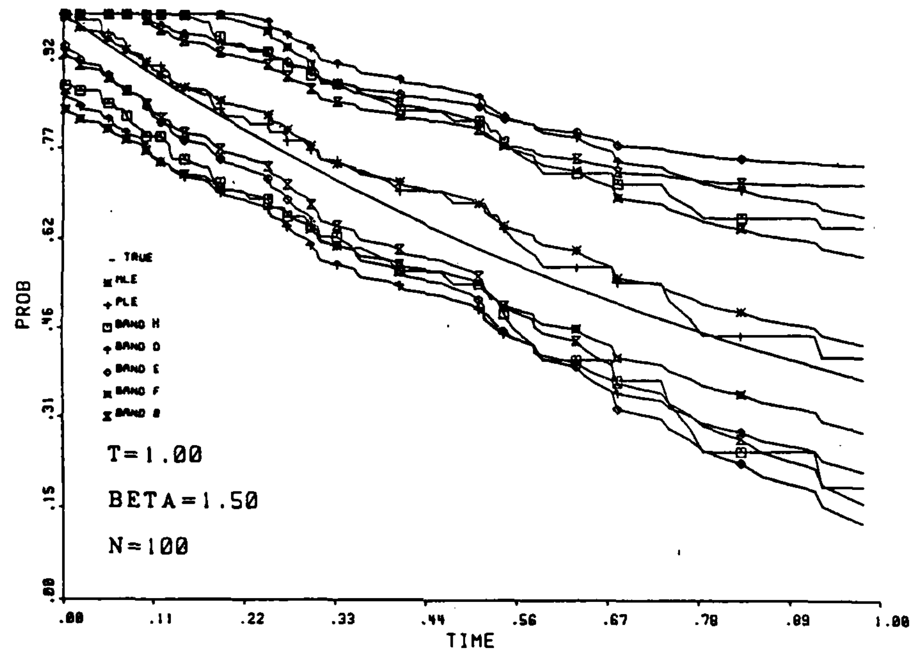
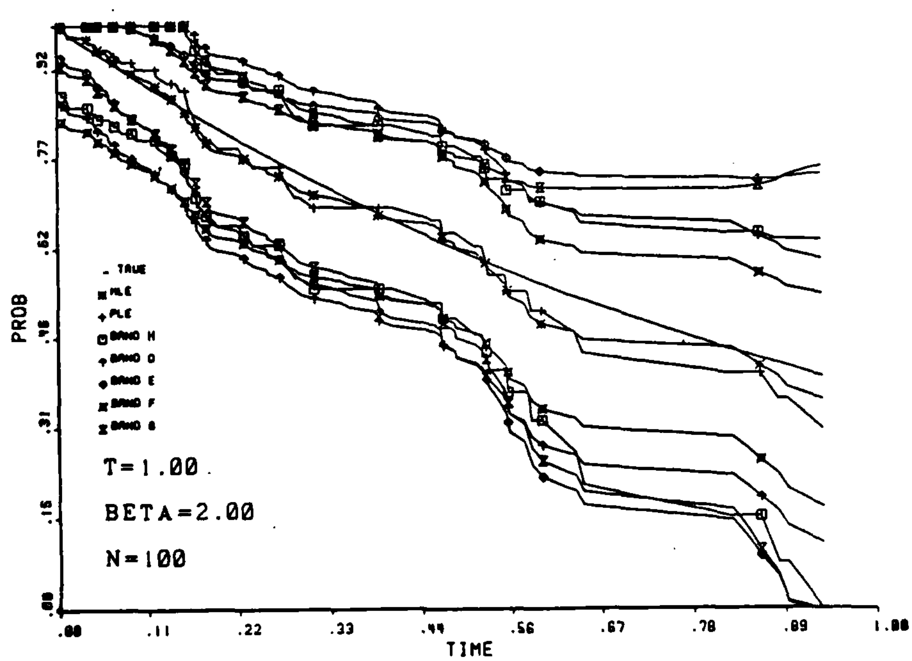


Figure 2d. Plots of 90% Confidence Bands for Randomly Generated Data Under the KG Model with $\beta = 2.0$



most situations we expect band $(FG)_n$ to be narrower than band H_n on almost the whole region of interest.

In Figures 2a-d we present superimposed plots of the MLE-based bands D_n^2, E_n^2, F_n^2 and G_n^2, and the Gillespie-Fisher PLE-based band H_n^2. These bands are asymptotic 90% simultaneous confidence bands for F based on four randomly generated sets of censored data each of size 100. The MLE and the PLE of F are also plotted. In plotting the bands and estimators via the Versatec plotter, we have taken the liberty of connecting adjacent points by straight lines. The four data sets were generated via the KG model considered in the simulation, with the censoring parameter β taking values 0.50, 1.00, 1.50 and 2.00. In constructing the bands, T was set equal to 1.0. Notice that when β is small the differences among the bands are not very apparent, but as β increases the differences among the bands are clear-cut. These plots are consistent with our analytical results showing the F_n- and D_n-bands are narrower for large t, and the G_n- and E_n-bands are narrower for small t. The small interval emanating from 0 where band E_n is narrower than band G_n is also visible in Figures 2a-d. Finally, under the KG model it is seen that the PLE-based band holds its own with the MLE-based bands when the censoring proportion is small (β=0.50), but as β increases it is easily outperformed by the MLE-based bands.

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